Math 4650
Homework 2
Solutions

(1)

Note that $\left|\frac{1}{\sqrt{n}} - 0\right| = \left|\frac{1}{\sqrt{n}}\right| \underset{\boxed{n \geq 1}}{=} \frac{1}{\sqrt{n}}$

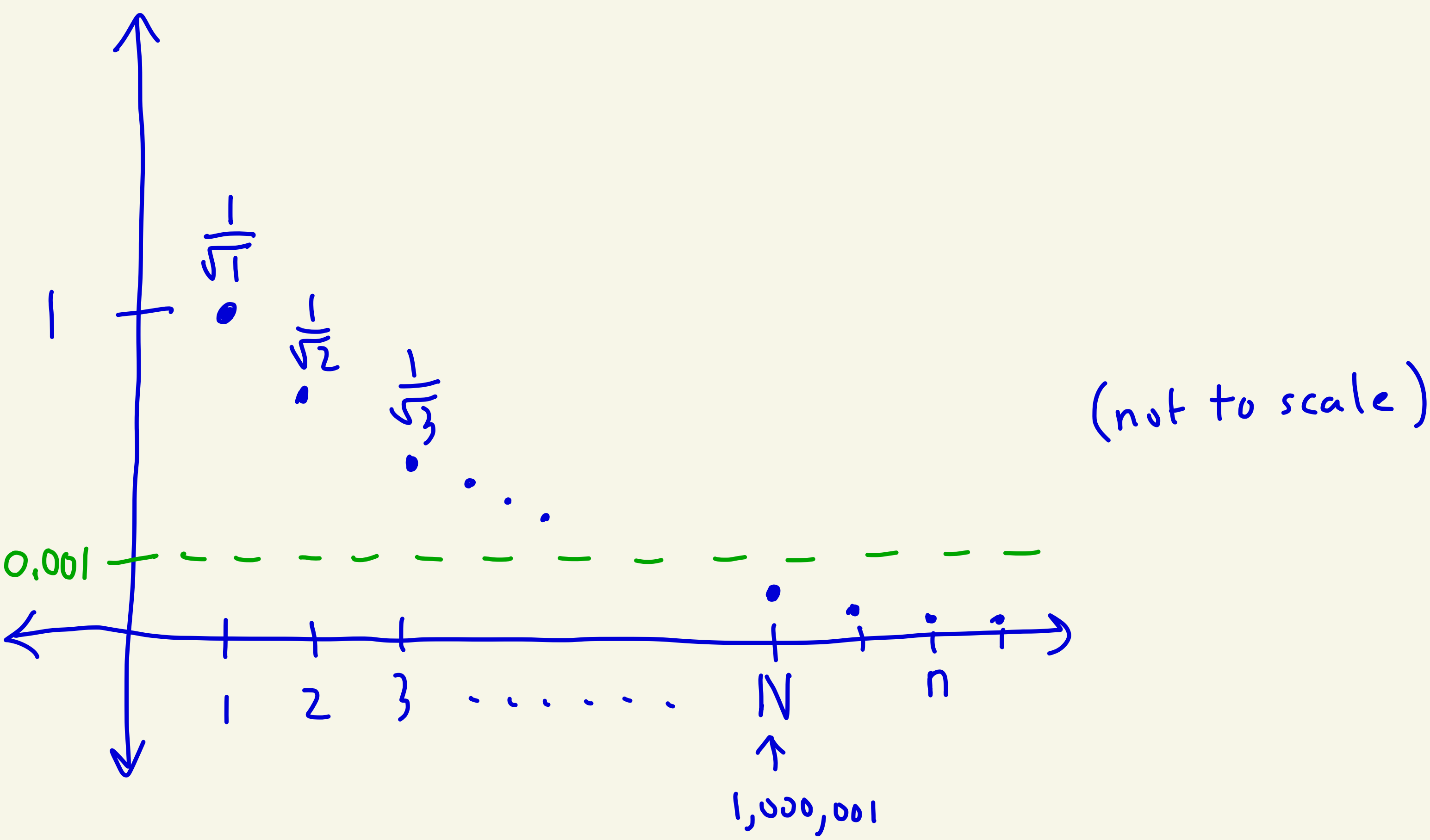
And $\frac{1}{\sqrt{n}} < 0.001$ iff $\frac{1}{0.001} < \sqrt{n}$

iff $\left(\frac{1}{0.001}\right)^2 < n$

iff $1{,}000{,}000 < n$

Let $N = 1{,}000{,}001$.

Then if $n \geq N$ we have $\left|\frac{1}{\sqrt{n}} - 0\right| < 0.001$



$\boxed{2}$

Note that $\left|\frac{2n}{3n+1} - \frac{2}{3}\right| = \left|\frac{6n-6n-2}{(3n+1)(3)}\right| = \left|\frac{-2}{9n+3}\right|$

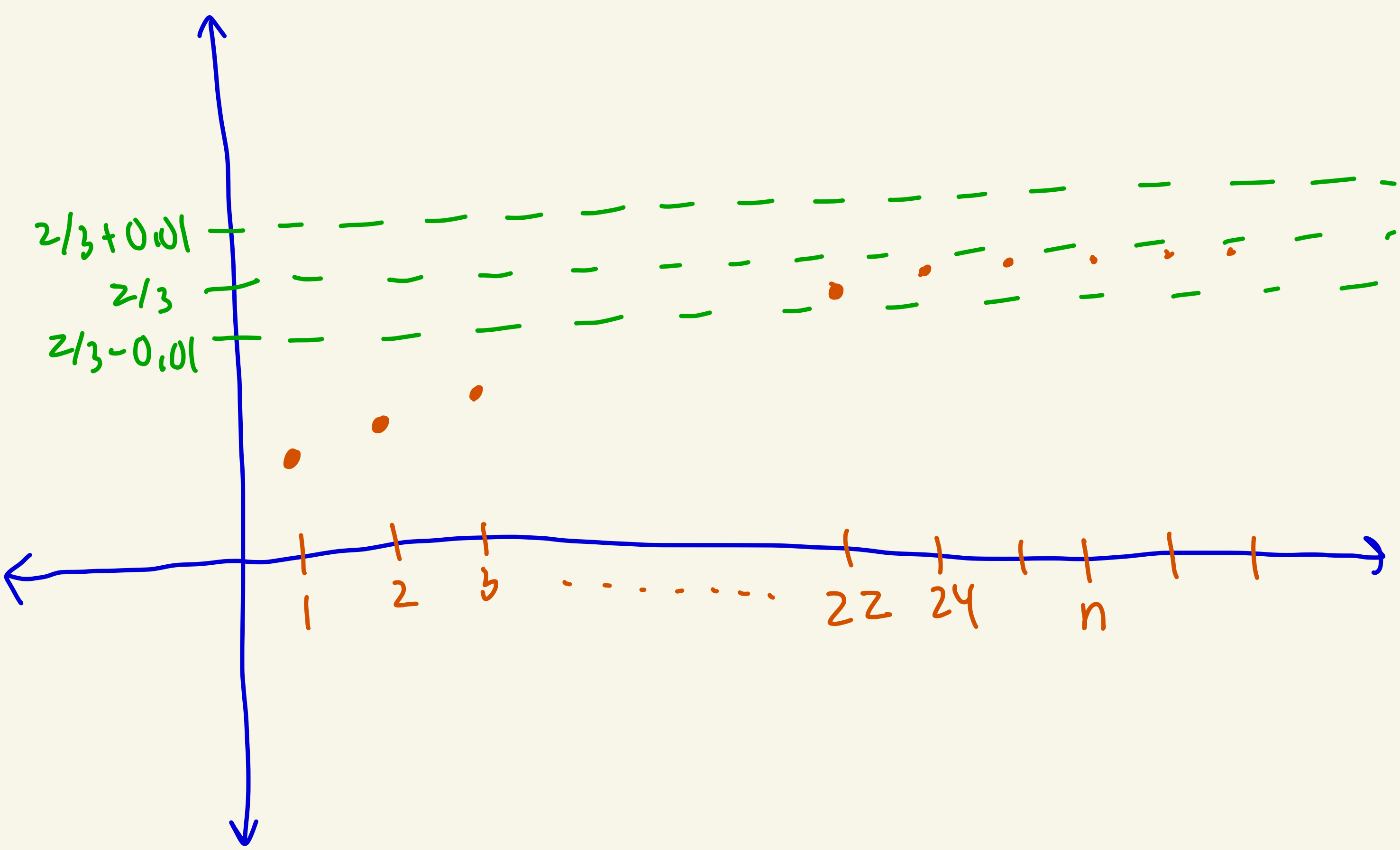
$$= \frac{|-2|}{|9n+3|} \underset{\boxed{n \geq 1}}{=} \frac{2}{9n+3}$$

And $\frac{2}{9n+3} < 0.01$ iff $\frac{2}{0.01} < 9n+3$ iff $\frac{\left(\frac{2}{0.01}-3\right)}{9} < n$

iff $21.89 < n$

Let $N = 22$.

Then if $n \geq N$ we have $\left|\frac{2n}{3n+1} - \frac{2}{3}\right| < 0.01$



$\boxed{3(a)}$

Let $\varepsilon > 0$.

Note that $\left|\frac{1}{\sqrt{n}} - 0\right| = \left|\frac{1}{\sqrt{n}}\right| \underset{n \geq 1}{=} \frac{1}{\sqrt{n}}$

We have that $\frac{1}{\sqrt{n}} < \varepsilon$ iff $\frac{1}{\varepsilon} < \sqrt{n}$ iff $\frac{1}{\varepsilon^2} < n$.

Let N be an integer with $N > \frac{1}{\varepsilon^2}$.

Then if $n \geq N$ we have $\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$

from above.

$\blacksquare$

3 (b)

Let $\varepsilon > 0$.

Note that

$$\left| \frac{n+2}{5n-3} - \frac{1}{5} \right| = \left| \frac{5n+10-5n+3}{(5n-3)(5)} \right|$$

$$= \left| \frac{13}{25n-15} \right| = \frac{13}{25n-15}$$

since $n \geq 1$
we know
$25n-15 \geq 25(1)-15 = 10 > 0$

And $\frac{13}{25n-15} < \varepsilon$ iff $\frac{13}{\varepsilon} < 25n-15$

iff $\frac{1}{25}\left(\frac{13}{\varepsilon}+15\right) < n$.

Let N be an integer with $N > \frac{1}{25}\left(\frac{13}{\varepsilon}+15\right)$

Then if $n \geq N$ we have $\left| \frac{n+2}{5n-3} - \frac{1}{5} \right| < \varepsilon$

from above.

$\blacksquare$

$\boxed{3(c)}$

Let $\varepsilon > 0$.

Note that

since $n+1 > n$ we know $\sqrt{n+1} > \sqrt{n}$ which gives $\sqrt{n+1} - \sqrt{n} > 0$

$$\left|\left(\sqrt{n+1} - \sqrt{n}\right) - 0\right| = \left|\sqrt{n+1} - \sqrt{n}\right| = \sqrt{n+1} - \sqrt{n}$$

$$= \frac{\left(\sqrt{n+1} - \sqrt{n}\right)}{1} \cdot \frac{\left(\sqrt{n+1} + \sqrt{n}\right)}{\left(\sqrt{n+1} + \sqrt{n}\right)}$$

$$= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$< \frac{1}{\sqrt{n} + \sqrt{n}} = \frac{1}{2\sqrt{n}}$$

$\sqrt{n+1} + \sqrt{n} > \sqrt{n} + \sqrt{n}$

So, $\dfrac{1}{\sqrt{n+1} + \sqrt{n}} < \dfrac{1}{\sqrt{n} + \sqrt{n}}$

And $\dfrac{1}{2\sqrt{n}} < \varepsilon$ iff $\dfrac{1}{\sqrt{n}} < 2\varepsilon$ iff $\dfrac{1}{2\varepsilon} < \sqrt{n}$

iff $\dfrac{1}{4\varepsilon^2} < n$.

Let N be an integer with $N > \dfrac{1}{4\varepsilon^2}$.

Then if $n \geq N$ we have

$\left|\left(\sqrt{n+1} - \sqrt{n}\right) - 0\right| < \varepsilon$ from above.

$\boxed{//}$

3(d)

We will show that $(n^4)_{n=1}^{\infty}$ is an unbounded sequence and thus $\lim_{n\to\infty} n^4$ does not exist.

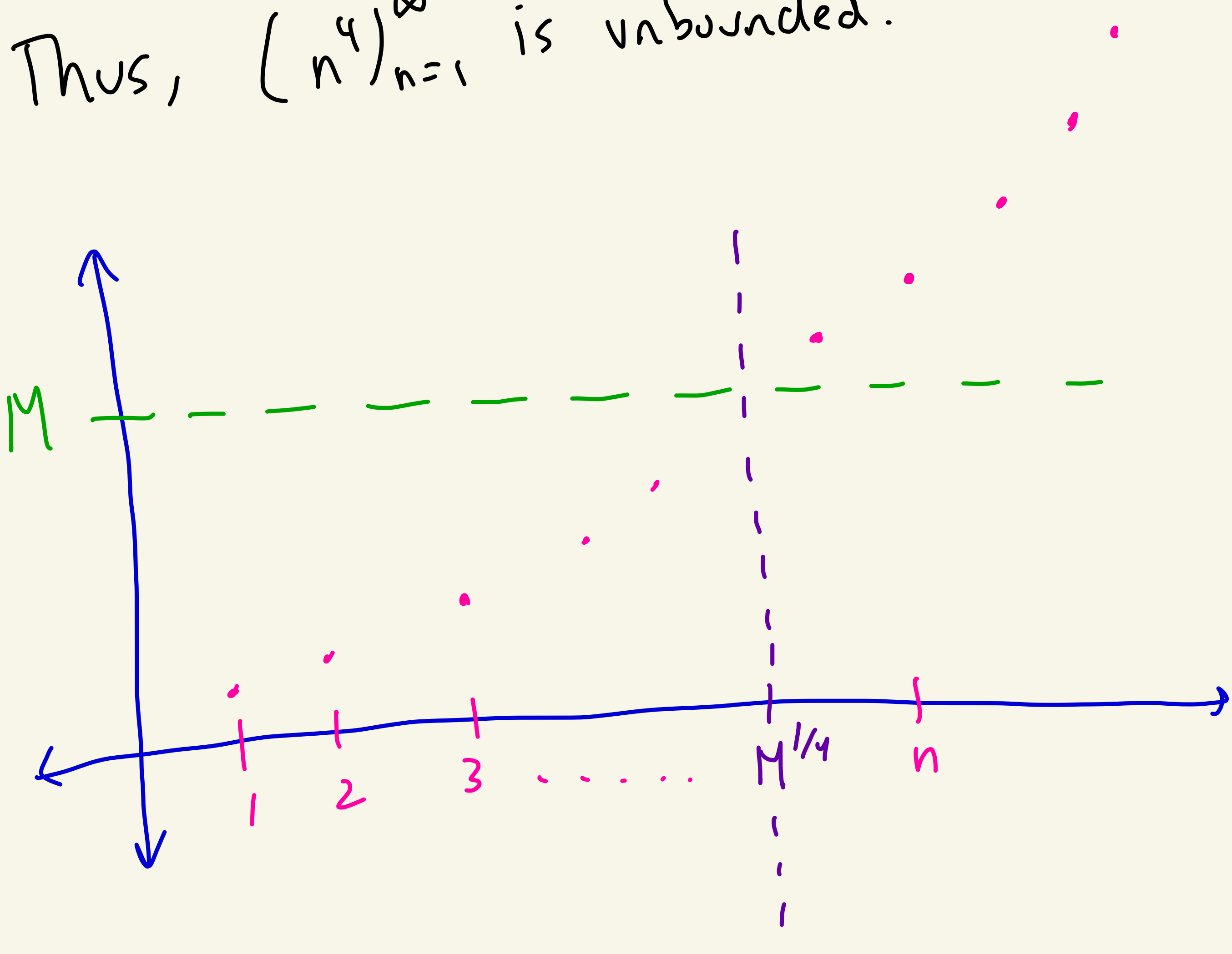
Let $M>0$ be any fixed number.

Then if $n > M^{1/4}$ we have

$$|n^4| = n^4 > (M^{1/4})^4 = M$$

So if $n > M^{1/4}$ then $n^4 > M$.

Thus, $(n^4)_{n=1}^{\infty}$ is unbounded.



3(e)

Let $\varepsilon > 0$.

Note that $$\left|\frac{n^2}{2n^2+1} - \frac{1}{2}\right| = \left|\frac{2n^2 - 2n^2 - 1}{(2n^2+1)(2)}\right|$$
$$= \left|\frac{-1}{4n^2+2}\right| = \frac{|-1|}{|4n^2+2|}$$
$$= \frac{1}{4n^2+2}$$

And $\frac{1}{4n^2+2} < \varepsilon$ iff $\frac{1}{\varepsilon} < 4n^2+2$ iff $\frac{1}{\varepsilon} - 2 < 4n^2$

iff $\frac{1}{4}\left(\frac{1}{\varepsilon} - 2\right) < n^2$ iff $\sqrt{\frac{1}{4}\left(\frac{1}{\varepsilon} - 2\right)} < n$

Let N be an integer with $N > \sqrt{\frac{1}{4}\left(\frac{1}{\varepsilon} - 2\right)}$.

Then if $n \geq N$ we have

$$\left|\frac{n^2}{2n^2+1} - \frac{1}{2}\right| < \varepsilon$$

from above.

3(f) Let $0 < r < 1$.

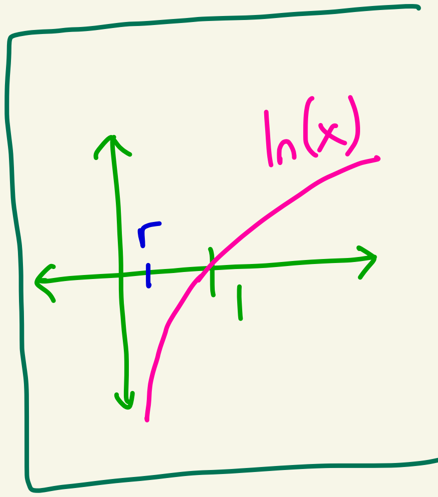
Let $\varepsilon > 0$

Then $|r^n - 0| = |r^n| = r^n$

$0<r<1$
So, $r^n > 0$

And $r^n < \varepsilon$ iff $\ln(r^n) < \ln(\varepsilon)$

iff $n \cdot \underbrace{\ln(r)}_{\substack{\text{negative} \\ \text{since } 0<r<1}} < \ln(\varepsilon)$



iff $n > \dfrac{\ln(\varepsilon)}{\ln(r)}$

inequality flips since we multiplied both sides by a negative #

Let N be an integer with
$$N > \frac{\ln(\varepsilon)}{\ln(r)}$$

Then if $n \geq N$ we have $|r^n - 0| < \varepsilon$ from above.

$\boxed{//}$

$\boxed{3(g)}$

Let $\varepsilon > 0$.

Note that

$$\left| \frac{\sqrt{n^2+1}}{n!} - 0 \right| = \left| \frac{\sqrt{n^2+1}}{n!} \right| = \frac{\sqrt{n^2+1}}{n!} \leq \frac{\sqrt{n^2+n^2}}{n!}$$

$\boxed{\frac{\sqrt{n^2+1}}{n!} > 0}$ $\boxed{1 \leq n^2}$

$$= \frac{\sqrt{2n^2}}{n!} = \frac{\sqrt{2} \cdot n}{n!} = \frac{\sqrt{2} \cdot n}{n \cdot (n-1)!}$$

$n! = n(n-1)!$

Ex: $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$6! = 6 \cdot [5!]$

$$= \frac{\sqrt{2}}{(n-1)!} \leq \frac{\sqrt{2}}{n-1}$$

$(n-1)! \geq n-1$

So, $\frac{1}{(n-1)!} \leq \frac{1}{n-1}$

Note that $\frac{\sqrt{2}}{n-1} < \varepsilon$ iff $\frac{\sqrt{2}}{\varepsilon} < n-1$ iff $\frac{\sqrt{2}}{\varepsilon} - 1 < n$.

Choose an integer N where $N > \frac{\sqrt{2}}{\varepsilon} - 1$.

Then if $n \geq N$ we get $\left| \frac{\sqrt{n^2+1}}{n!} - 0 \right| < \varepsilon$ from above. $\boxed{///}$

$\boxed{4(a)}$ Let $\varepsilon > 0$.

Since $a_n \to A$ we know there exists $N > 0$ where if $n \geq N$ then $|a_n - A| < \frac{\varepsilon}{|\alpha|}$.

Then if $n \geq N$ we get

$$
\begin{aligned}
|\alpha a_n - \alpha A| &= |\alpha(a_n - A)| \\
&= |\alpha| \cdot |a_n - A| \\
&< |\alpha| \cdot \frac{\varepsilon}{|\alpha|} \\
&= \varepsilon.
\end{aligned}
$$

Thus, if $n \geq N$ then $|\alpha a_n - \alpha A| < \varepsilon$

Hence $\alpha a_n \to \alpha A$. $\boxed{//}$

$\boxed{4(b)}$ Let $\varepsilon > 0$.

Since $a_n \to A$ there exists N_1 where if $n \geq N_1$ then $|a_n - A| < \frac{\varepsilon}{2|\alpha|}$

Since $b_n \to B$ there exists N_2 where if $n \geq N_2$ then $|b_n - B| < \frac{\varepsilon}{2|\beta|}$

Let $N = \max\{N_1, N_2\}$.

Thus, if $n \geq N$, then

$$|(\alpha a_n + \beta b_n) - (\alpha A + \beta B)| = |(\alpha a_n - \alpha A) + (\beta b_n - \beta B)|$$

$$\leq |\alpha a_n - \alpha A| + |\beta b_n - \beta B|$$

$$= |\alpha(a_n - A)| + |\beta(b_n - B)|$$

$$= |\alpha||a_n - A| + |\beta||b_n - B|$$

$$< |\alpha| \cdot \frac{\varepsilon}{2|\alpha|} + |\beta| \cdot \frac{\varepsilon}{2|\beta|}$$

$$= \varepsilon$$

Thus, if $n \geq N$, then $|(\alpha a_n + \beta b_n) - (\alpha A + \beta B)| < \varepsilon$

So, $\alpha a_n + \beta b_n \longrightarrow \alpha A + \beta B$.

$\boxed{//}$

4(c) Let $\varepsilon > 0$.

Note that

$$|a_n b_n - AB| = |a_n b_n - b_n A + b_n A - AB|$$

$$\leq |a_n b_n - b_n A| + |b_n A - AB|$$

$$= |b_n| \cdot |a_n - A| + |A| \cdot |b_n - B|$$

Since (b_n) converges it is bounded so there exists $M > 0$ where $|b_n| \leq M$ for all n.

Since $a_n \to A$ there exists $N_1 > 0$ where

if $n \geq N_1$ then $|a_n - A| < \frac{\varepsilon}{2M}$

Since $b_n \to B$ there exists $N_2 > 0$ where

if $n \geq N_2$ then $|b_n - B| < \frac{\varepsilon}{2(|A|+1)}$

$|A|+1$ is used in case $|A| = 0$

Let $N = \max\{N_1, N_2\}$

Then if $n \geq N$ we get that

$$
\begin{aligned}
|a_n b_n - AB| &\leq |b_n| \cdot |a_n - A| + |A| \cdot |b_n - B| \\
&< M \cdot \frac{\varepsilon}{2M} + |A| \cdot \frac{\varepsilon}{2(|A|+1)} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \underbrace{\left(\frac{|A|}{|A|+1} \right)}_{\text{this is less than } 1} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

So if $n \geq N$, then $|a_n b_n - AB| < \varepsilon$.

Hence $a_n b_n \to AB$. $\blacksquare$

5 Let $\varepsilon > 0$.

Since $a_n \to L$ there exists $N_1 > 0$ where

if $n \geq N_1$ then $|a_n - L| < \varepsilon$.

So, $\boxed{\text{if } n \geq N_1 \text{ then } -\varepsilon < a_n - L < \varepsilon.}$

Since $c_n \to L$ there exists $N_2 > 0$ where

if $n \geq N_2$ then $|c_n - L| < \varepsilon$.

So, $\boxed{\text{if } n \geq N_2 \text{ then } -\varepsilon < c_n - L < \varepsilon.}$

Let $N = \max\{N_1, N_2\}$.

Then if $n \geq N$ we have that

$$-\varepsilon < a_n - L \underset{a_n \leq b_n}{\leq} b_n - L \underset{b_n \leq c_n}{\leq} c_n - L < \varepsilon$$

So if $n \geq N$ then $-\varepsilon < b_n - L < \varepsilon$.

So if $n \geq N$ then $|b_n - L| < \varepsilon$.

Hence $b_n \to L$.

$\blacksquare$

6(a) Suppose $x_n \geq 0$ for all n and $\lim_{n\to\infty} x_n = L$.

Let's show that $L \geq 0$.

Suppose to the contrary that $L < 0$.

Let $\varepsilon = |L| > 0$.

Since $x_n \to L$, there exists an integer $N > 0$ where if $n \geq N$ then $|x_n - L| < \underbrace{|L|}_{\varepsilon}$.

In particular $|x_N - L| < |L|$.

But then $L - |L| < x_N < L + |L|$.

Since $L < 0$ we know $|L| = -L$.

Thus, $2L < x_N < 0$.

But then $x_N < 0$ which is a contradiction since $x_n \geq 0$ for all n.

$$|A| = \begin{cases} A & \text{if } A \geq 0 \\ -A & \text{if } A < 0 \end{cases}$$

Hence $L \geq 0$. ▨

6(b) Suppose $a_n \leq b_n$ for all n and $a_n \to A$ and $b_n \to B$.

Then, $0 \leq b_n - a_n$ for all n and $b_n - a_n \to B - A$.

By part (a), we get $B - A \geq 0$.

Thus, $B \geq A$. ▨

$\boxed{⑥(c)}$

By part (b) since $C \leq a_n$ for all n we know that $\lim_{n\to\infty} C \leq \lim_{n\to\infty} a_n$.

Thus, $C \leq \lim_{n\to\infty} a_n$

Also by part (b) since $a_n \leq D$ for all n we know that $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} D$.

Thus, $\lim_{n\to\infty} a_n \leq D$.

Hence, $C \leq \lim_{n\to\infty} a_n \leq D$.

7 (METHOD 1)

Let $\varepsilon = \frac{|L|}{2} > 0$ ← $\frac{|L|}{2} > 0$ since $L \neq 0$

Since $a_n \to L$ there exists $N > 0$ where if $n \geq N$ then $|a_n - L| < \frac{|L|}{2}$.

Thus, if $n \geq N$ then
$$\begin{aligned} |L| &= |L - a_n + a_n| \\ &\leq |L - a_n| + |a_n| \\ &< \frac{|L|}{2} + |a_n|. \end{aligned}$$

So if $n \geq N$, then $|L| < \frac{|L|}{2} + |a_n|$

So if $n \geq N$, then $\frac{|L|}{2} < |a_n|$.

Let $M = \frac{|L|}{2} > 0$.

Then $|a_n| > M$ for all $n \geq N$. $\blacksquare$

(METHOD 2) Let $\varepsilon = \frac{|L|}{2} > 0$. ← $\frac{|L|}{2} > 0$ since $L \neq 0$

Since $a_n \to L$ there exists $N > 0$ where if $n \geq N$ then $|a_n - L| < \frac{|L|}{2}$.

Thus if $n \geq N$ then
$$\big|\,|a_n| - |L|\,\big| \underset{\uparrow \text{ HW 1}}{\leq} |a_n - L| < \frac{|L|}{2}$$

So if $n \geq N$ then
$$-\frac{|L|}{2} < |a_n| - |L| < \frac{|L|}{2}$$

So if $n \geq N$, then

$$\frac{|L|}{2} < |a_n| < \frac{3|L|}{2}$$

Let $M = \frac{|L|}{2} > 0$.

Then if $n \geq N$ we have $|a_n| > M$. $\blacksquare$

8

Let (a_n) be a sequence with $a_n \to L$.

Let (a_{n_k}) be a subsequence.

Let $\varepsilon > 0$.

Since $a_n \to L$ there exists $N > 0$ where if $n \geq N$ then $|a_n - L| < \varepsilon$.

Thus if $n_k \geq N$, then $|a_{n_k} - L| < \varepsilon$.

So, $a_{n_k} \to L$. $\blacksquare$

9

Let $\varepsilon = 1$.
Since (a_n) is Cauchy there exists an integer $N > 0$ where if $n, m \geqslant N$ then $|a_n - a_m| < 1$.

So, if $n \geqslant N$ then $|a_n - a_N| < 1$.

So, if $n \geqslant N$ then

$$|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N| < 1 + |a_N|$$

Let

$$M = \max\{\underbrace{|a_1|, |a_2|, |a_3|, \ldots, |a_{N-1}|}_{\text{bounds for } |a_1|, |a_2|, \ldots, |a_{N-1}|}, \underbrace{1 + |a_N|}_{\text{bound for } |a_n| \text{ where } n \geq N}\}$$

Then $|a_n| \leq M$ for all n. $\boxed{//}$